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Bejan's constructal theory of equal potential distribution

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Abstract

Bejan's constructal theory is explored for a paradigm problem in electrical theory, analogue of a wide range of transport problems involving flow subject to a Ohm's Law model. The equi-potential optimisation is derived and generalised to further functional behaviour of resistances. The original case illustrates the power of Bejan's asymptotic-intersection method.

Keywords: Shape optimisation; Constructal theory; Intersecting asymptotes

1. Introduction

In Adrian Bejan's recent book on constructal theory [1], a theme emerges that when two differing regimes of transport are optimised, the available potential drop may be shared equally between the two. For example, we have the pressure drop across regions of diffusive flow and of turbulent flow, or the temperature drop across regions of high and low conductivity. Indeed the heat transfer example is how constructal theory was first reported, in this journal [2]. This note illustrates and generalises the essence of his result in a simple example; the principles hold for any transport phenomenon. The present example is worded in terms of electrical currents and resistances which seems both a new application of Bejan's constructal theory as well as being entirely familiar to engineers and scientists. Electrical analogies are, of course, a well-known source of techniques in heat transfer. The example itself has been deliberately simplified so as to illustrate a teaching point rather than provide a solution to any specific problem. The same arguments apply to any model where a potential drop is proportional to a current flow, and further generalisations are given towards the end of the note. The note finishes with an illustration taken from this model of Bejan's asymptotic intercept approximation.

2. The model

Consider the geometric optimisation of Fig. 1 where a horizontal strip of high conductivity material is available to pass a current through the system that otherwise consists of low conductivity material. The volume of the low resistance material is smaller than that of the high resistance material by a ratio r say. The maximum overall resistance will be between the opposite corner points where the current enters and where it leaves. The problem posed is to minimise the voltage difference between these two points necessary to drive the current, by adjusting the shape of the system while leaving the amount of the two components fixed.

As Bejan says, this maximum potential is indeed for the source corner furthest away from the sink and it will be sufficient therefore to consider some quantum I_0 injected in this region. Given the assumed proportionality of voltage drop and current, the problem is equally one of maximising the current for a given potential drop.

The variable available is the slenderness ratio, $s = Y/X$; actual amounts of the two materials are fixed in this two-dimensional problem with a system with unit thickness in the third dimension. The upper strip has the volume $\mathrm{Vol}_{\mathrm{upper}} = XY$ and the lower some fraction of

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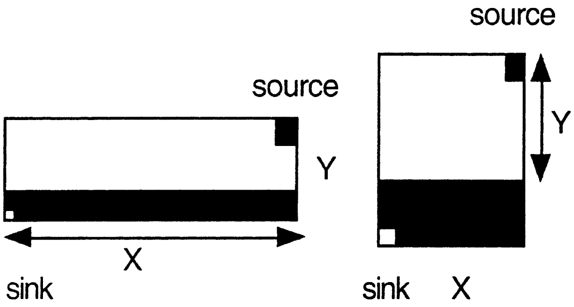

Fig. 1. Two configurations for the high and low conductivity strips with electrical current from the source to the sink. The available material is fixed so that $XY = \text{constant}$.

this such that the volume ratio is $r = \text{Vol}_{\text{below}}/\text{Vol}_{\text{above}}$. The lower strip has a resistivity ρ_{lo} smaller than that of the upper strip ρ_{hi}. For both strips the effective resistance is between one side and an opposing corner. The strips will have a resistance $R_{\text{sq}} = \rho S$ where $S = S(X, Y)$ is a *shape* factor. If $X = Y = 1$ then $S_{\text{sq}} = S(1, 1)$ will be approximately unity and certainly the same for both upper and lower regions. The resistance of each strip depends however on its orientation. For the lower strip where the current of interest is horizontal, the resistance falls with cross-sectional thickness A and rises with length in the x-direction. For the upper strip, the current of interest is vertical and similarly its resistance falls with cross-section A and rises with length in the y-direction. Note that the fixed amount of material means that as X is extended, Y decreases in proportion for both strips.

We can assume therefore that the resistance of the strips is of the form

$$R_{\text{lo}}(X, Y) \sim \rho_{\text{lo}} S_{\text{sq}} X/Y \quad \text{and} \quad R_{\text{hi}}(X, Y) \sim \rho_{\text{hi}} S_{\text{sq}} Y/X \tag{1}$$

Since both regions are rectangular, they will have a *common* shape factor and its value would not affect any internal optimisation. For simplicity put the common square shape scaling factor to unity $S_{\text{sq}} = 1$. (If $S_{\text{sq}} \neq 1$, then write an effective resistivity $\rho_{\text{eff}} = \rho S_{\text{sq}}$.) These resistances can be written therefore as

$$R_{\text{lo}} = \rho_{\text{lo}} \frac{X}{A_{\text{lo}}} = \rho_{\text{lo}} \frac{X}{rY} \quad \text{and} \quad R_{\text{hi}} = \rho_{\text{hi}} \frac{Y}{A_{\text{hi}}} = \rho_{\text{hi}} \frac{Y}{X} \tag{2}$$

We next assume an Ohm's Law situation such that the current (whether of electricity, heat, fluid flow, matter, etc.) is given by the potential drop divided by the resistance: $I_0 = \Delta V/R$ where the current I_0 is fixed.

3. Optimisation

We may now write the individual potential drops in series passing the common current and we have to minimise $\nabla V = \nabla V_{\text{above}} + \nabla V_{\text{below}}$. We use the Lagrange method [3] to optimise over X, Y writing the restraint between these two variables explicitly in the Lagrangian by introducing the fixed upper volume $\text{Vol} = XY$

$$L = \Delta V + \lambda[\text{Vol}_{\text{above}} - XY] \tag{3}$$

Then at an optimum

$$\frac{\partial L}{\partial X} = 0 = I_0 \left[\frac{\rho_{\text{lo}}}{rY} - \frac{\rho_{\text{hi}} Y}{X^2} \right] - \lambda Y \tag{4}$$

and

$$\frac{\partial L}{\partial Y} = 0 = I_0 \left[\frac{\rho_{\text{hi}}}{X} - \frac{\rho_{\text{lo}}}{rY^2} \right] - \lambda X \tag{5}$$

Eliminating the Lagrange multiplier λ gives

$$\frac{X^2}{Y^2} = r \frac{\rho_{\text{hi}}}{\rho_{\text{lo}}} \quad \text{or} \quad X^2_{\text{opt}} = \text{Vol}_{\text{above}} \sqrt{r \frac{\rho_{\text{hi}}}{\rho_{\text{lo}}}} \tag{6}$$

whence

$$\Delta V_{\text{below}} = I_0 \sqrt{\frac{\rho_{\text{hi}} \rho_{\text{lo}}}{r}} = \Delta V_{\text{above}} \tag{7}$$

That is, the available potential is divided equally in the optimum arrangement where

$$\left(\frac{Y}{X} \right)_{\text{opt}} = \left(\frac{\rho_{\text{lo}} \text{Vol}_{\text{above}}}{\rho_{\text{hi}} \text{Vol}_{\text{below}}} \right)^{1/2} \tag{8}$$

This may be interpreted as having the optimum slenderness ratio the reciprocal of the square root of an *effective* volume ratio.

The attraction of the Lagrange multiplier route to optimisation is not only the replacement of a constrained optimisation by a free optimisation but also the interpretation of the Lagrange multiplier as a measure of the effect of a small change of restraint on the optimum. In this case the Lagrange multiplier is zero, showing indirectly that there is no change of potential drop on rescaling the geometry, at this optimum slenderness ratio. This can also be seen from the results for ΔV_{opt} of course, given fixed current. (If the current were proportional to the size, however, as in a distributed loading, the potential would go up accordingly.)

4. Generalised equi-potential

The equi-partition remains true even if resistances take the form $\rho(X/Y)^n$ with n arbitrary (put $(X/Y)^n = x/y$). To see this, suppose that the two resistances are given by the forms $As^m + Bs^{-n}$ where $s = Y/X$ is the slenderness ratio. This reduces to the original form when $m = n = 1$. This may be optimised directly in terms of the ordinary differential variable s to give

$$\frac{\mathrm{d}}{\mathrm{d}s}As^m + Bs^{-n} = 0 = mAs^{m-1} - nBs^{-n-1}$$
$$\text{or} \quad s^{m+n} = \frac{nB}{mA} \tag{9}$$

and the optima

$$\nabla V_{\text{below}} = A^{n/(m+n)}B^{m/(m+n)}\left(\frac{n}{m}\right)^{m/(m+n)} \quad \text{and}$$
$$\nabla V_{\text{above}} = A^{n/(m+n)}B^{m/(m+n)}\left(\frac{n}{m}\right)^{n/(m+n)} \tag{10}$$

so that these two are equal, with an equal division of potential drop between the two components, whenever $m = n$. More generally the ratio is as $(m/n)^{n/m}$.

We see therefore that for a simple case of two dissimilar components 'in series' the optimum arrangement of given material to achieve a given purpose calls for an equal division of the potential drops available to drive the system. This theme is elegantly illustrated in much of Bejan's writing where he derives optimal shapes for areas from this principle, extending to entire flow structures such as tree networks.

5. Method of intersecting asymptotes

The original model also provides an instructive example of Bejan's intersecting-asymptotes method to approximate an optimum design and its associated values. In the expression for the potential drop $\Delta V = \Delta V_{\text{above}} + \Delta V_{\text{below}}$ we have two terms, one going as the slenderness ratio $s = Y/X$ and one as its reciprocal $s^{-1} = X/Y$. We may readily optimise this as an ordinary differential in s to find as before

$$\frac{\mathrm{d}}{\mathrm{d}s}\Delta V = \frac{\mathrm{d}}{\mathrm{d}s}I_0\left[\frac{\rho_{\text{lo}}}{rs} + \rho_{\text{hi}}s\right] = 0 = \rho_{\text{hi}} - \frac{\rho_{\text{lo}}}{rs^2} \tag{11}$$

so that

$$s_{\text{opt}}^2 = \frac{\rho_{\text{lo}}\text{Vol}_{\text{above}}}{\rho_{\text{hi}}\text{Vol}_{\text{below}}} \tag{12}$$

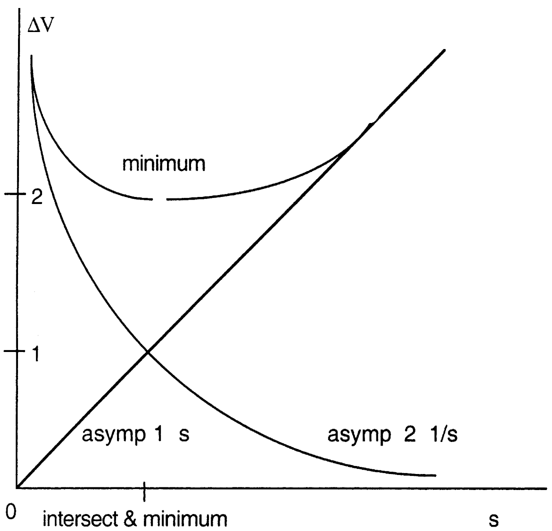

Fig. 2. The approximate fit of a dependent variable to its asymptotes, showing the agreement with the intersection of asymptotes and an optimum that is double the intersect.

If the two separate terms are then plotted against s as asymptotic terms, Fig. 2, then we see that these two asymptotes intersect at a value of the independent argument s that is indeed the optimum value and that the true optimum value of ΔV is indeed twice the optimum value of each, corresponding to equal contributions from both asymptotic terms. This result remains true for any common power of s as already shown.

More generally there may be further terms contributing to a result than the two extreme asymptotes but their intersection is likely to be a fair estimate of the true optimised independent variable and twice the intersection value a passable estimate of the optimised dependent variable when the exact functional dependence is unknown.

6. Conclusion

We have provided a model based on optimising electrical currents that serves as analogy for a wide range of transport phenomena. It is now readily seen how Bejan's constructal theory leads to an equi-potential division between two competing regimes. Indeed a more general result is given when the functionality is made more general. The model has also illustrated the intersecting-asymptote method. In addition to providing a didactic model, it is hoped the extension of Bejan's theory to electrical circuits will have intrinsic value.

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